Section 5-2, Mathematics 104

Review Laws of Exponents

$a^n a^m = a^{n+m}$	$\dfrac{a^n}{a^m} = a^{n-m}$	for $a \neq 0 \quad a^0 = 1$
$\dfrac{1}{a^n} = a^{-n}$	$\left(a^n\right)^m = a^{nm}$	$\left(ab\right)^n = a^n b^n$
$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$	$\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$	$\dfrac{a^{-n}}{b^{-m}} = \dfrac{b^m}{a^n}$

Example:

$$\left(x^2 y^4\right)\left(3x\right) =$$

$$-2\left(y^2\right)^3 =$$

$$\frac{14a^5b^3}{7a^2b^2} =$$

$$\frac{x^n y^{3n}}{x^2 y^4} =$$

Polynomials

A **monomial** is an expression of the form ax^n where a is a constant, x is a variable, and n is a natural number. n is sometimes called the **degree** of the polynomial.

A **polynomial** is an expression formed from the sum of monomials, eg.

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

Each monomial in the expression is called a **term.**

The a_i's are constants, with a_0 called the **constant term.**

n the largest exponent of the variable is **degree** of the polynomial.

You should already be familiar with first degree polynomials such as

$$5x + 7$$

and second degree polynomials such as

$$3x^2 - 2x + 8$$

One thing that will make polynomials important, is that many real world problems distill down to an equation:

$P(x) = 0$, where $P(x)$ is a polynomial.

We will learn how to solve this type of equation in a few different ways.

Adding Polynomials

When adding and subtracting polynomials, we match up **like** terms, which are terms with the same degree. This can be done either horizontally or vertically

Example

$$\left(2x^2+4x-1\right)+\left(x^2-3\right)=$$
$$2x^2+x^2+4x-1-3=$$
$$3x^2+4x-4$$

Example

$$\left(5x^4+3x^2-2x+1\right)+\left(4x^3+x^2+3x+5\right)=$$

$$\begin{array}{|cccccc}
5x^4 & & 3x^2 & -2x & 1 \\
+ & 4x^3 & x^2 & 3x & 5 \\
5x^4 & 4x^3 & 4x^2 & x & 6
\end{array}$$

Subtracting Polynomials

Subtracting polynomials is the same as adding, just make sure to distribute the negative sign properly

$$\left(2x^2+3\right)-\left(3x^2-4\right)=$$
$$2x^2+3-3x^2+4=$$
$$-x^2+7$$

Multiplying Polynomials

To multiply polynomials we use the distributive law.

Example:

$$\left(3x^2+1\right)\left(x^3+x\right)=3x^2\left(x^3+x\right)+1\left(x^3+x\right)=3x^5+3x^3+x^3+x=3x^5+4x^3+x$$

You can do this type of multiplication the same way you would do multiplication:

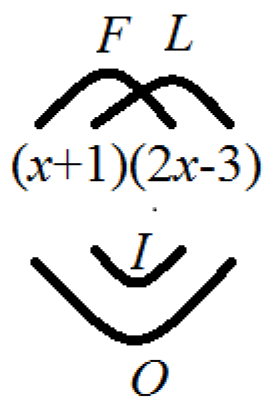
Example:

$$\begin{array}{c|ccccc}
 & & & 2x^2 & -x & 4 \\
 & \times & & & 2x & -3 \\
 & & & -6x^2 & 3x & -12 \\
 & + & 4x^3 & -2x^2 & 8x & \\
 & & 4x^3 & -8x^2 & 11x & -12
\end{array}$$

FOIL

Foil is a way to multiply two first degree polynomials.

Foil stands for First, Outside, Inside and Last
$(x+1)(2x-3)=[F]2x^2-[O]3x+[I]2x-[L]3=2x^2-x-3$



This is an important skill that we will use a lot in multiplying two linear polynomials is called FOIL.

Example:

$(x+3)(2x-5)$

The first's are x and $2x$ so we have $2x^2$

The inner and outer usually combine. They are

inner: 3 times $2x$ = $6x$

and

outer x times -5 = $-5x$

so combined we just have x

last's are 3 and -5 = -15

so we end up with

$(x+3)(2x-5)=2x^2+5x-15$

Factoring Polynomials

Just as we can factor integers, eg.

$48 = 2\times2\times2\times2\times3 = 2^4\times3$ (Note these are prime factors

Polynomials can be factored, eg.

$x^2+5x+6=(x+3)(x+2)$

Important Patterns (you need to memorize)

There are a number of ways of factoring polynomials. To start with, there are some important patterns that you should be familiar with:

$$A^2+2AB+B^2=(A+B)^2$$
$$A^2-2AB+B^2=(A-B)^2$$
$$A^2-B^2=(A+B)(A-B)$$

A good exercise is to FOIL the expressions on the right.

It is not always obvious but you will need to learn to look for these patterns.

Example:

$4x^2+12x+9=(2x)^2+2x\cdot3+(3)^2=(2x+3)^2$

or

$9x^2-1=(3x)^2-1^2=(3x+1)(3x-1)$

Polynomials of the form A^2+B^2 are **irreducible**, meaning they can't be factored.

Irreducible polynomials are like prime numbers.

They have no factors.

Factoring 2nd degree polynomials

If you don't immediately see one of these patterns, your first instinct to factor a 2nd degree polynomial should be using **reverse foil**.

To do reverse foil, break up and write a potential factoring

$x^2 - 5x + 6 = (x?2)(x?3)$ and see what signs, + or - will make it work by using foil. In this case we see that two negative signs will work

$(x-2)(x-3) = x^2 - 2x - 3x + 6 = x^2 - 5x + 6$

When dealing with constants with more than one set of factors, trial and error can take some time.

Example:
$4x^2 + 16x + 15$

Since 15 isn't a perfect square we can't use one of our patterns.

Both 4 and 15 have factors 2 x 2 and 3 x 5

We don't have to worry about the signs since they both must be +

There are a number of possible combinations we can check:

$(4x+1)(x+15)$
$(4x+3)(x+5)$
$(4x+5)(x+3)$
$(2x+3)(2x+5)$

We have to FOIL these to check but we've already set things up so the first's and last come out right. So we just have to add the inner and outer and see which one adds up to 16

1+60 no
20+3 no
4+15 no
6+10 YES!

When there are negative signs, things get trickier, but there are obvious patterns.

Example:

$x^2 - x - 6$

You should always make sure the first has a positive coefficient.
If it doesn't factor -1 out of all terms.

Note that if the last has a negative sign, then there must be one + and one -

Possible solutions are:

$(x+1)(x-6)$
$(x-1)(x+6)$
$(x+3)(x-2)$
$(x-3)(x+2)$

Again we check the inner and outer coefficients, looking for -1

1+-6 no
6+-1 no
3+-2 no
-3+2 Yes

Note that if the constant term is positive and the coefficient of x is negative, then you must have two minus signs. Why?

Factoring using patterns and reverse FOIL does not always work!

Factoring using patterns and reverse FOIL does not always work!

The quadratic formula

When all else fails, the quadratic formula will always enable you to factor a trinomial, or discover that it is irreducible.

The quadratic formula works like this.

If $ax^2+bx+c=(x-x_1)(x-x_2)$ where we can calculate x_1 *and* x_2 using the formula

$$x_1=\frac{-b+\sqrt{b^2-4ac}}{2a}$$
$$x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

We write the quadratic formula this way:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Note the part of the formula under the square root sign b^2-4ac

This is called **the discriminant.**

If the discriminant is less than zero, you have the square root of a negative number.
In this case the trinomial cannot be factored.

If the discriminant is zero, x_1 *and* x_2 are the same and you have the pattern
$A^2\pm 2AB+B^2=(A\pm B)^2$

Otherwise you factor as $(x-x_1)(x-x_2)$

Example:

$x^2+x-1=?$

$$x=\frac{-1\pm\sqrt{1+4}}{2}=\frac{-1\pm\sqrt{5}}{2}$$
$$x^2+x-1=\left(x-\frac{-1+\sqrt{5}}{2}\right)\left(x-\frac{-1-\sqrt{5}}{2}\right)$$

Section 5-4

Dividing Polynomials

We have two ways of dividing polynomials

Long Division

and

Synthetic Division

Long Division always works.

If you know long division of numbers, long division of polynomials should come quite easily.

Example:

$$\frac{2x^2+3x+4}{x+2}$$

$\begin{array}{rl} & 2x \\ x+2 & \overline{)2x^2+3x+4} \end{array}$	$\begin{array}{rl} & 2x \\ x+2 & \overline{)2x^2+3x+4} \\ & \underline{2x^2+4x} \\ & \qquad -x+4 \end{array}$
$\begin{array}{rl} & 2x \\ x+2 & \overline{)2x^2+3x+4} \\ & 2x^2+4x \end{array}$	$\begin{array}{rl} & 2x-1 \\ x+2 & \overline{)2x^2+3x+4} \\ & \underline{2x^2+4x} \\ & \qquad -x+4 \\ & \qquad \underline{-x-2} \\ & \qquad\qquad 6 \end{array}$

$$\frac{2x^2+3x+4}{x+2} = 2x-1+\frac{6}{x+2}$$

or
$2x$ - 1 Remainder 6

Synthetic Division

Synthetic division does not do anything better than long division.
It only works if you are dividing by x-a where a is some constant.
It is also a strange looking skill that you have to learn carefully.
So why learn synthetic division?

Because once you learn it is easy.

Here's the previous example using synthetic division

$$\frac{2x^2+3x+4}{x+2}$$

x+2 = x-(-2)

$$\begin{array}{r|rrr} -2 & 2 & 3 & 4 \\ & & -4 & 2 \\ \hline & 2 & -1 & 6 \end{array}$$

The first two numbers are coefficients

2x-1 and the last number is the remainder